$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ \lim\limits_{t \to 0} u(x,t) = 0 \end{cases}$$

Clearly, $u(x,t) = 0$ is a solution.

$u(x,t) := \frac{x}{t} \mathcal{H}_t(x)$ where $\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is the heat kernel.

1 (a) u satisfies the heat equation.

$$\frac{\partial u}{\partial t} = -\frac{x}{t^2} \mathcal{H}_t(x) + \frac{x}{t} \left(\frac{\partial}{\partial t} \mathcal{H}_t(x) \right)$$

$$= -\frac{x}{t^2} \mathcal{H}_t(x) + \frac{x}{t} \left(-\frac{1}{2t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^2} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \right)$$

$$= \mathcal{H}_t(x) \left(-\frac{x}{t^2} - \frac{x}{2t^2} + \frac{x^3}{4t^3} \right).$$

$$\frac{\partial u}{\partial x} = \frac{1}{t}\mathcal{H}_t(x) + \frac{x}{t}\frac{\partial}{\partial x}\left(\mathcal{H}_t(x)\right)$$

$$= \frac{1}{t}\mathcal{H}_t(x) + \frac{x}{t}\left(-\frac{x}{2t}\mathcal{H}_t(x)\right)$$

$$= \mathcal{H}_t(x)\left(\frac{1}{t} - \frac{x^2}{2t^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\mathcal{H}_t(x)\left(\frac{1}{t} - \frac{x^2}{2t}\right) + \mathcal{H}_t(x)\left(-\frac{x}{t^2}\right)$$

$$= \mathcal{H}_t(x)\left(-\frac{x}{2t}\right)\left(\frac{1}{t} - \frac{x^2}{2t^2}\right) + \mathcal{H}_t(x)\left(-\frac{x}{t^2}\right)$$

$$= \mathcal{H}_t(x)\left(-\frac{x}{2t^2} - \frac{x^3}{4t^3} - \frac{x}{t^2}\right)$$

$$= \frac{\partial u}{\partial t}.$$

(b) $\lim\limits_{t\to 0^+} u(x,t) = 0$ for any $x \in \mathbb{R}$.

Pf: $$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \frac{x}{t}\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

$\left(y = \frac{1}{t}\right)$ $$= \lim_{y\to\infty} xy\frac{\sqrt{y}}{\sqrt{4\pi}} e^{-\frac{x^2}{4}y}$$

$$= \lim_{y\to\infty} \frac{x}{\sqrt{4\pi}} y^{\frac{3}{2}} e^{-\frac{x^2}{4}y}$$

When $x=0$, $\frac{x}{\sqrt{4\pi}} y^{\frac{3}{2}} e^{-\frac{x^2}{4}y} = 0$, $\forall\, y>0$.

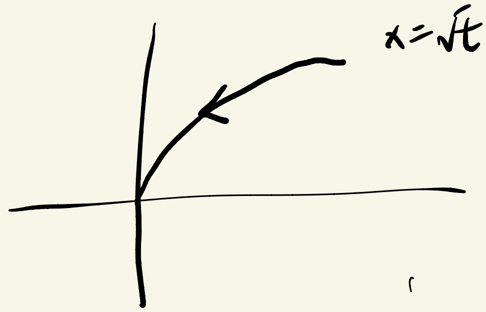
When $x\neq 0$, $e^{\frac{x^2}{4}y}$ grows faster than $y^{\frac{3}{2}}$.

Then $\lim_{y\to\infty} \frac{x}{\sqrt{4\pi}} y^{\frac{3}{2}} e^{-\frac{x^2}{4}y} = 0$

Hence $\lim_{t\to 0^+} u(x,t) = 0$.

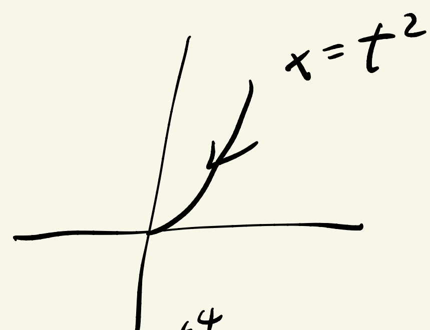
(c) u is not continuous at $(0,0)$.

Proof: • We put $x=\sqrt{t}$ and choose the path $(\sqrt{t}, t) \to (0,0)$



$$\lim_{(\sqrt{t},t)\to(0,0)} u(x,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{t}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4}} = \infty .$$

• On the other hand, we put $x=t^2$ and choose the path $(t^2, t) \to (0,0)$



$$\lim_{(t^2,t)\to(0,0)} u(x,t) = \lim_{t\to 0} \frac{t^2}{t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{t^4}{4t}}$$

$$= \lim_{t \to 0} \sqrt{\frac{t}{4\pi}} e^{-\frac{t^3}{4}} = 0$$

Hence u is continuous at $(0,0)$.

$\square$

2 Suppose f is continuous function of moderate decrease such that

$$\int_{-\infty}^{\infty} f(y) e^{-y^2} e^{2xy} dy = 0 \quad \text{for any } x \in \mathbb{R}.$$

Then $f = 0$.

Pf: Let $g(z) := e^{-z^2}$

$$0 = \int_{-\infty}^{\infty} f(y) e^{-y^2} e^{2xy} dy$$

$$= e^{x^2} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2} dy$$

$$= e^{x^2} f * g(x).$$

Since $e^{x^2} \neq 0$, $\forall x \in \mathbb{R}$, then

$f*g(x) = 0$, $\forall x \in \mathbb{R}$.

Then $\hat{f}(\xi)\hat{g}(\xi) = \widehat{f*g}(\xi) = 0$, $\forall \xi \in \mathbb{R}$.

$g(x) = e^{-x^2}$, $e^{-\pi \delta x^2} \xrightarrow{\mathcal{F}} \frac{1}{\sqrt{\delta}} e^{-\pi \frac{\xi^2}{\delta}}$

Take $\delta = \frac{1}{\pi}$, $\hat{g}(\xi) = \sqrt{\pi}\, e^{-\pi^2 \xi^2} \neq 0$, $\forall \xi \in \mathbb{R}$.

Then $\hat{f}(\xi) = 0$, $\forall \xi \in \mathbb{R}$.

Since $f, \hat{f} \in \mathcal{M}(\mathbb{R})$, by Fourier Inversion Formula, $f(x) = 0$, $\forall x \in \mathbb{R}$.

3 Let $h(x) := e^{-|x|} \cos x$.

Then $\hat{h}(\xi) = 2 \frac{(2\pi\xi)^2 + 2}{(2\pi\xi)^4 + 4}$.

Compute $\int_{-\infty}^{\infty} \left(\frac{x^2+2}{x^4+4} \right)^2 dx$.

Pf: Recall $f(\lambda x) \xrightarrow{\mathcal{F}} \frac{1}{\lambda} \hat{f}\left(\frac{\xi}{\lambda}\right)$

Let $g(x) = \pi\, h(2\pi x) = \pi e^{-2\pi|x|} \cos 2\pi x$

$$\hat{g}(x) = \cancel{\frac{2\pi}{2\pi}} \frac{\xi^2+2}{\xi^4+4}$$

$$\int_{-\infty}^{\infty} \left(\frac{x^2+2}{x^4+4} \right)^2 dx = \int_{-\infty}^{\infty} |\hat{g}(x)|^2 d\xi$$

(Plancherel formula) $$= \int_{-\infty}^{\infty} |g(x)|^2 dx$$

$$= \pi^2 \int_{-\infty}^{\infty} e^{-4\pi|x|} (\cos 2\pi x)^2 \, dx$$

$$= 2\pi^2 \int_0^{\infty} e^{-4\pi x} \left(\frac{e^{2\pi i x} + e^{-2\pi i x}}{2} \right)^2 dx$$

$$= \frac{\pi^2}{2} \int_0^{\infty} e^{-4\pi x} \left(e^{4\pi i x} + 2 + e^{-4\pi i x} \right)$$

$$= \frac{\pi^2}{2} \int_0^{\infty} \left(2e^{-4\pi x} + e^{(-4\pi + 4\pi i)x} + e^{(-4\pi - 4\pi i)x} \right) dx$$

$$= \frac{\pi^2}{2} \left(-\left(\frac{2}{-4\pi} \right) - \left(\frac{1}{-4\pi + 4\pi i} \right) - \left(\frac{1}{-4\pi - 4\pi i} \right) \right)$$

$$= \frac{\pi^2}{2} \left(\frac{2}{4\pi} + \frac{1}{4\pi - 4\pi i} + \frac{1}{4\pi + 4\pi i} \right)$$

$$= \frac{\pi^2}{2} \left(\frac{1}{2\pi} + \frac{8\pi}{16(\pi^2 + \pi^2)} \right)$$

$$= \frac{\pi^2}{2} \left(\frac{1}{2\pi} + \frac{1}{4\pi} \right)$$

$$= \frac{3\pi}{8}. \qquad \square$$